Theoretical Physics III (Quantum Mechanics)

Sample Exam — Duration: 150 min – Max. Points: 100

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1. Quantum Mechanics in 2-dimensional Hilbert space (3+7+3+7 = 20 Points)

Consider two observables **A** and **B** and a state $|\psi\rangle$, which in a fixed basis of the two-dimensional Hilbert space can be represented as follows:

$$\mathbf{A} \doteq \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \qquad \mathbf{B} \doteq \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix} \qquad |\psi\rangle \doteq c \begin{pmatrix} 1 \\ i \end{pmatrix} .$$

(a) Calculate the normalization constant $c > 0$.

(b) What are the possible values when measuring **A** ? With what probability can they be measured in the state $|\psi\rangle$?

(c) Determine the expectation values of **B** in the state $|\psi\rangle$.

(d) Calculate explicitly the time evolution of a state $|\chi\rangle_t$ with $|\chi\rangle_{t=0} = |\psi\rangle$ for time $t > 0$, under the assumption that $\eta\mathbf{B}$ is the Hamiltonian of the system ($\eta \in \mathbb{R}$ has a dimension of energy).

2. Harmonic Oscillator (5+5+10+10 = 30 Points)

The Hamiltonian of a particle of mass m in a one-dimensional harmonic oscillator potential and the associated lowering operator are given by

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \qquad \text{and} \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\sqrt{\frac{1}{2m\omega\hbar}}\,\hat{p}\,.$$

(a) Calculate $[a, a^\dagger]$ using the known canonical commutation relation.

(b) Give the Hamiltonian in momentum representation.

(c) Consider in this part of the exercise the Hamiltonian of the shifted harmonic oscillator $H' = H + \lambda\hat{x}$ with any $\lambda \in \mathbb{R}$.
- Determine the exact energy eigenvalues.
- Calculate the energy shift in any eigenstate of H in the first-order of the perturbation theory with respect to the term $\lambda\hat{x}$ and compare with the exact result.

(d) At time $t = 0$ a state $|\chi\rangle_t$ is described by the wave function of the ground state $|0\rangle$ (with $\langle x|0\rangle = \varphi_0(x)$), that is spatially shifted by $b \in \mathbb{R}$, so the following applies $\langle x|\chi\rangle_{t=0} = \varphi_0(x - b) = \langle x - b|0\rangle$.
- Calculate the expectation value $\langle\hat{x}\rangle_{|\chi\rangle_t}$ for $t > 0$. Note that we are dealing with the harmonic potential.
- Write the state $|\chi\rangle_{t=0}$ in the form of $|\chi\rangle_{t=0} = \hat{T}|0\rangle$ with a suitable operator $\hat{T}$. Then specify the state $|\chi\rangle_t$ for arbitrary time $t > 0$.

Hint:

- *For two operators $\hat{A}$ and $\hat{B}$ that commute with the commutator $[\hat{A}, \hat{B}]$ the following holds $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}$.*
- *For an arbitrary operator $\hat{A}$, $\exp(\hat{A}) = \sum_{n=0}^{\infty} \hat{A}^n/n!$ holds.*

3. The symmetric top (8+12 = 20 Points)

Consider the Hamiltonian of a symmetric top

$$H = \frac{\hat{j}_1^2 + \hat{j}_2^2}{A} + \frac{\hat{j}_3^2}{B} ,$$

where $\hat{\vec{j}}$ is the angular momentum operator with integer j and $A, B \neq 0$, $A \neq B$ represent real constants.

(a) Determine the spectrum of H, the eigenstates and their degeneracy.

(b) At time $t = 0$ a state $|j = 1, m_1 = 1\rangle$ is given, where m_1 is the magnetic quantum number associated with the $\hat{j}_1$ component.

- Express the operator $\hat{j}_1$ in the basis $\{|j, m_3\rangle\}$ given by $\hat{\vec{j}} \cdot \hat{\vec{j}}$ and $\hat{j}_3$.
- Calculate the time evolution of the state $|j = 1, m_1 = 1\rangle$ for $t > 0$.

Hint: The following relation holds: $\hat{j}_\pm|j, m_3\rangle = \hbar\sqrt{j(j+1) - m_3(m_3 \pm 1)}\,|j, m_3{\pm}1\rangle$.

4. Particles in one dimension (8+10+6+6 = 30 Points)

Consider a particle of mass m, which moves in the one-dimensional potential

$$V(x) = \begin{cases} \infty & \text{for } -a/2 > x \\ V_0\delta(x) & \text{for } -a/2 < x < a/2 \\ \infty & \text{for } \qquad\quad x > a/2 \end{cases}$$

with $V_0 \geq 0$, $a > 0$.

(a) Consider first the case $V_0 = 0$. Calculate the eigenvalues and normalized eigenvectors of the Hamiltonian and thus determine the general solution of the time-dependent Schödinger equation.

Hint: You may use that the following holds: $\int_0^{n\pi} \sin^2(x)\, dx = \pi n/2$, $n = 1, 2, \ldots$.

(b) Show that for $V_0 > 0$ the energy E of the system can be expressed by the following implicit equations $\sin(ka/2) = 0$ or $\tan(ka/2) = -\hbar^2 k/(mV_0)$ with $k = \sqrt{2mE/\hbar^2}$. Discuss E in the limit $V_0 \to 0$ and compare with task part (a).

(c) Determine with the help of (b) the degree of degeneracy of the energy eigenvalues for $V_0 > 0$. Calculate in the simplest way, the expectation value of the position operator in any eigenstate of the Hamiltonian.

(d) Calculate for all eigenstates of the Hamiltonian with $V_0 = 0$ the induced energy shift by the term $V_0\delta(x)$ in the first order of the perturbation theory.